THOMAS WHITHAM SIXTH FORM

GCSE Further Mathematics

Revision Guide

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The book contains a number of worked examples covering the topics needed in the Further Mathematics Specifications. This includes Calculus, Trigonometry, Geometry and the more advanced Algebra.

Algebra

- **<u>Indices for all rational exponents</u>**

$$a^m \times a^n = a^{m+n} \;,\; a^m \div a^n = a^{m-n} \;,\; \left(a^m\right)^n = a^{mn} \;,\; a^0 = 1 \;,$$

$$a^{\frac{1}{2}} = \sqrt{a} \;,\; a^{\frac{1}{n}} = \sqrt[n]{a} \;,\; a^{-n} = \frac{1}{a^n} \;,\; \frac{1}{a^{-n}} = a^n \;,$$

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

<u>Example</u> $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

<u>Example</u> $9^{-\frac{1}{2}} = \frac{1}{9^{\frac{1}{2}}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$

<u>Example</u> $x^{\frac{2}{3}} = 4 \Rightarrow \left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = 4^{\frac{3}{2}} \Rightarrow x = \left(\sqrt{4}\right)^3 = \underline{8}$

<u>Example</u> Simplify $2x^{-1} \times (3x)^2 \div 6x^3$

$$2x^{-1} \times (3x)^2 \div 6x^3 = \frac{2}{x} \times 9x^2 \times \frac{1}{6x^3} = \frac{18x^2}{6x^4} = \underline{\frac{3}{x^2}}$$

- **<u>Quadratic equations</u>** $ax^2 + bx + c = 0$

-

<u>Examples</u> (i) $4x^2 = 9$ (ii) $4x^2 = 9x$ (iii) $4x^2 = 9x - 2$

(i)

$$x^2 = \frac{9}{4}$$

$$\therefore x = \pm\frac{3}{2}$$

(ii)

$$4x^2 - 9x = 0$$

$$x(4x - 9) = 0$$

$$\therefore x = 0 \quad \text{or} \quad x = \frac{9}{4}$$

(iii)

$$4x^2 - 9x + 2 = 0$$

$$(4x - 1)(x - 2) = 0$$

$$\therefore x = 2 \quad \text{or} \quad x = \frac{1}{4}$$

Example Solve (i) $2x^2 + x - 5 = 0$ using the method of CTS

(ii) $5x^2 - 7x + 1 = 0$ using the formula.

(i) $2x^2 + x - 5 = 0$

$$x^2 + \frac{x}{2} = \frac{5}{2}$$

$$x^2 + \frac{x}{2} + \left(\frac{1}{4}\right)^2 = \frac{5}{2} + \left(\frac{1}{4}\right)^2$$

$$\left(x + \tfrac{1}{4}\right)^2 = \tfrac{5}{2} + \tfrac{1}{16}$$

$$\left(x + \tfrac{1}{4}\right)^2 = \tfrac{41}{16}$$

$$x + \tfrac{1}{4} = \pm\sqrt{\tfrac{41}{16}}$$

$$x = -\tfrac{1}{4} \pm \sqrt{\tfrac{41}{16}}$$

$$\underline{\underline{x \approx 1.35, \quad -1.85}}$$

- **Simultaneous equations**

Example Solve simultaneously $2x + 3y = 8, \quad y = x^2 - x + 2$

Here we substitute for *y* from the second equation into the first

$$2x + 3y = 8 \quad \therefore \quad 2x + 3\left(x^2 - x + 2\right) = 8 \quad \therefore \quad 3x^2 - x - 2 = 0$$

$$\therefore \quad (3x + 2)(x - 1) = 0 \qquad \therefore \quad x = -\tfrac{2}{3}, \quad 1$$

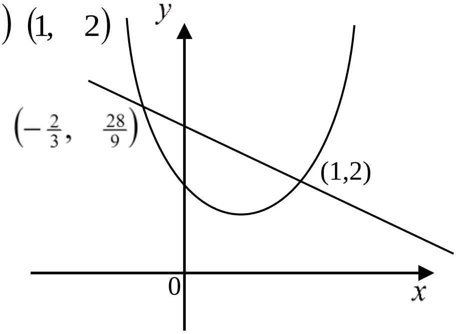
when $x=-\frac{2}{3}$, $y=\frac{4}{9}+\frac{2}{3}+2=\frac{28}{9}$

when $x=1$, $y=1-1+2=2$

Solutions, $x=-\frac{2}{3},\quad y=\frac{28}{9}$

$x=1,\quad y=2$

The geometrical interpretation here is that the straight line $2x+3y=8$ and the parabola $y=x^2-x+2$ intersect at points $\left(-\frac{2}{3},\ \frac{28}{9}\right)$ $(1,\ 2)$



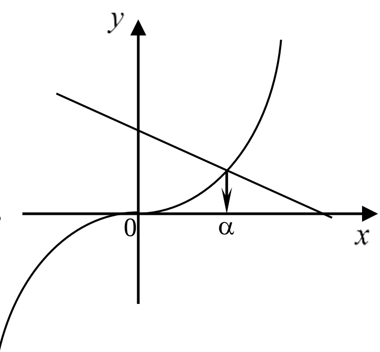
- **<u>Intersection points of graphs to 'solve' equations</u>**. There are many equations which can not be solved analytically. Approximate roots to equations can be found graphically if necessary.

<u>Example</u> What straight line drawn on the same axes as the graph of $y=x^3$ will give the real root of the equation $x^3+x-3=0$?

$x^3+x-3=0 \;\Rightarrow\; x^3=3-x$

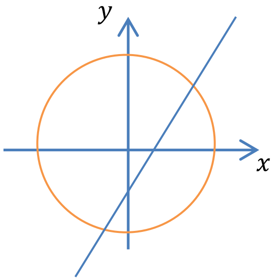
$\therefore$ draw $y=3-x$

As can be seen from the sketch there is only one real root α.



<u>Example</u>

Obtain the points of intersection of the circle $x^2 + y^2 = 10$ and the line $y = 2x + 1$



$$x^2 + y^2 = 10$$

$$x^2 + (2x + 1)^2 = 10$$

$$x^2 + 4x^2 + 4x + 1 = 10$$

$$5x^2 + 4x - 9 = 0$$

$$(5x + 9)(x - 1) = 0$$

$$\therefore \quad x = -\frac{9}{5} \quad , \; x = 1$$

Using $y = 2x + 1$ $\quad y = -\frac{7}{2} \quad , \; y = 3$

Hence points are $\left(-\frac{9}{2}, -\frac{7}{2}\right) \quad , \quad (1,3)$

- **<u>Expansions and factorisation –extensions</u>**

  <u>Example</u>

$$(2x-1)(x^2 - x + 3) = 2x^3 - 2x^2 + 6x$$

$$-x^2 + x - 3 \qquad \text{Expanding}$$

$$= \underline{2x^3 - 3x^2 + 7x - 3}$$

Example $x^3 - 9x = x(x^2 - 9) = \underline{x(x-3)(x+3)}$ Factorising

- **The remainder Theorem**

If the polynomial $p(x)$ be divided by $(ax+b)$ the remainder will be $p\left(-\frac{b}{a}\right)$

Example When $p(x) = 2x^3 - 3x - 5$ is divided by $2x+1$ the remainder is $p\left(-\frac{1}{2}\right) = -\frac{1}{4} + \frac{3}{2} - 5 = \underline{\underline{-\frac{15}{4}}}$

Example Find the remainder when $4x^3 - 3x^2 + 11x - 2$ is divided by $x - 1$.

$f(1) = 4(1)^3 - 3(1)^2 + 11(1) - 2 = 10$ $\therefore$ remainder is 10

- **The factor theorem** Following on from the last item

$p\left(-\frac{b}{a}\right) = 0 \Rightarrow ax + b$ is a factor of $p(x)$

Example Show that $(x-2)$ is a factor of $6x^3 - 13x^2 + x + 2$ and hence solve the equation $6x^3 - 13x^2 + x + 2 = 0$

Let $p(x) = 6x^3 - 13x^2 + x + 2$

$$p(2) = 6(2)^3 - 13(2)^2 + (2) + 2 = 48 - 52 + 2 + 2 = 0$$

$\therefore$ $(x-2)$ is a factor of $p(x)$

$p(x) = (x-2)(6x^2 - x - 1)$ ...by inspection

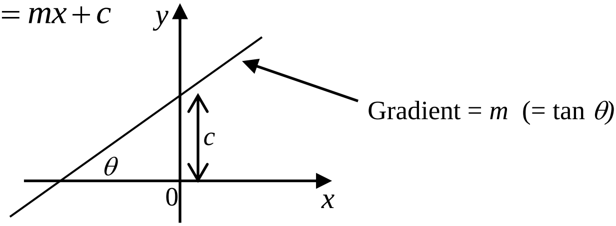
$$= (x-2)(3x+1)(2x-1)$$

$\therefore$ Solutions to the equation are $\underline{\underline{x = 2,\ -\frac{1}{3},\ \frac{1}{2}}}$

Geometry

- **<u>Gradient/ intercept form of a straight line</u>** Equation

$y = mx + c$



- **<u>Distance between two points</u>**

Given A$(x_1, \quad y_1)$ B$(x_2, \quad y_2)$ then

$$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

- **<u>Gradient of a line through two points</u>**... $A(x_1, \quad y_1)$ and $B(x_2, \quad y_2)$ say

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- **<u>Equation of a line through (*x'*, *y'*) of gradient *m*</u>**

$$y - y' = m(x - x')$$

- **<u>Equation of a line through two points</u>**

Find the gradient using $\underline{\underline{m = \frac{y_2 - y_1}{x_2 - x_1}}}$ and use the formula as above.

- **<u>Parallel and perpendicular lines</u>**

Let two lines have gradients m_1 and m_2

Lines parallel $\Leftrightarrow m_1 = m_2$

Lines perpendicular $\Leftrightarrow m_1 m_2 = -1$ or $m_1 = -\frac{1}{m_2}$

- **<u>Mid-point</u>** of line joining … $A(x_1, \quad y_1)$ and $B(x_2, \quad y_2)$ coordinates are

$$\underline{\left\{\tfrac{1}{2}(x_1 + x_2), \quad \tfrac{1}{2}(y_1 + y_2)\right\}}$$

- **<u>General form of a straight line</u>**

$ax + by + c = 0$. To find the gradient, rewrite in gradient/intercept form.

<u>Example</u> Given points $A(-2,\quad 3)$ and $B(1,\quad -1)$ find

(a) distance *AB*

(b) the coordinates of the mid-point *M* of *AB*

(c) the gradient of *AB*

(d) the equation of the line through $C(5,\quad 2)$ parallel to *AB*

(a) $AB^2 = (1+2)^2 + (-1-3)^2 = 9+16 = 25$ $\quad \therefore$ <u>*AB* = 5</u>

(b) $\underline{\underline{M\left(-\tfrac{1}{2},\quad 1\right)}}$

(c) Gradient *AB* = $\dfrac{-1-3}{1+2} = \underline{\underline{-\dfrac{4}{3}}}$

(d) Point $(5,\quad 2)$ Gradient = $-\frac{4}{3}$

<u>Equation</u> $y-2 = -\frac{4}{3}(x-5)$

$$3y-6 = -4x+20$$

$$\underline{\underline{3y+4x = 26}}$$

<u>Example</u> Find the gradient of the line $2x+3y=12$ and the equation of a perpendicular line through the point $(0,\quad -4)$

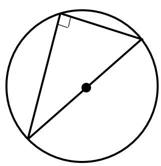
$2x + 3y = 12 \quad \Rightarrow 3y = -2x + 12 \qquad \Rightarrow y = -\tfrac{2}{3}x + 4$

$\therefore$ gradient = $\underline{\underline{-\tfrac{2}{3}}}$

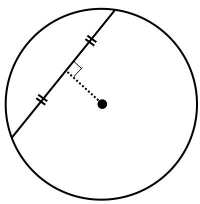
Gradient of perpendicular = $-\dfrac{1}{-\tfrac{2}{3}} = \underline{\underline{\tfrac{3}{2}}}$

<u>Equation</u> $\qquad \underline{\underline{y = \tfrac{3}{2}x - 4}} \qquad (y = mx + c)$

- **<u>The Circle</u>**



Angles in semicircle is 90°



Perpendicular to a chord from centre of circle bisects the chord.

- **<u>Centre, radius form of equation</u>**

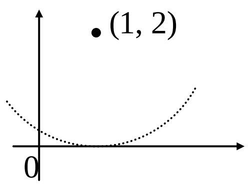
$$(x-a)^2 + (y-b)^2 = r^2$$

Centre (*a, b*) radius = *r*

<u>Example</u> Centre (2, -1) radius 3 <u>equation</u> $(x-2)^2 + (y+1)^2 = 9$

Example Centre (1, 2) touching $0x$ Equation

$$(x-1)^2+(y-2)^2=4$$



- **General form of equation**

$$(x-a)^2+(y-b)^2=r^2$$ Circle centre (a,b) with radius r

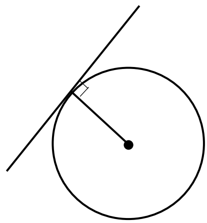
To find centre and radius, use the method of CTS to change into centre/radius form.

Example $x^2+y^2-2x+3y-3=0$

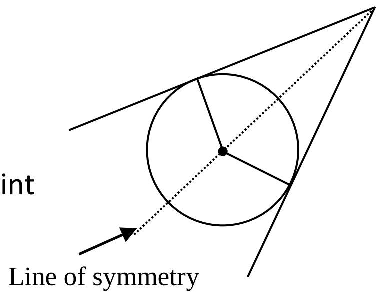
$$x^2+y^2-2x+3y-3=0$$
$$(x^2-2x)+(y^2+3y)=3$$
$$(x^2-2x+1)+\left(y^2+3y+\left(\tfrac{3}{2}\right)^2\right)=3+1+\left(\tfrac{3}{2}\right)^2$$
$$(x-1)^2+\left(y+\tfrac{3}{2}\right)^2=\tfrac{25}{4}$$

$\therefore$ Centre $\left(1,-\frac{3}{2}\right)$ radius = $\frac{5}{2}$

❑ **Tangents**



Angle between tangent and radius drawn to point of contact is 90°



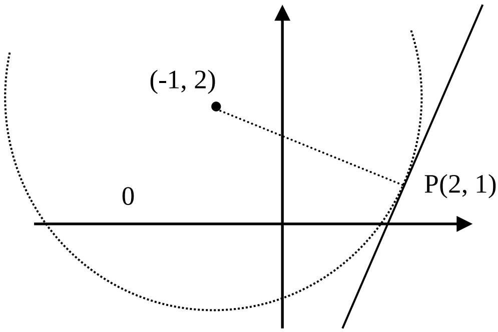
Tangents drawn from extended point

Example Find the equation of the tangent to the circle

$x^2 + y^2 + 2x - 4y - 5 = 0$ at the point P(2, 1)

$$x^2 + y^2 + 2x - 4y - 5 = 0$$
$$\left(x^2 + 2x\right) + \left(y^2 - 4y\right) = 5$$
$$\left(x^2 + 2x + 1\right) + \left(y^2 - 4y + 4\right) = 5 + 1 + 4$$
$$(x+1)^2 + (y-2)^2 = 10$$

$\therefore$ Centre at $(-1,\ 2)$, radius $\sqrt{10}$



Gradient CP = $\dfrac{2-1}{-1-2} = -\dfrac{1}{3}$

$\therefore$ gradient of tangent at P = 3

<u>Equation</u>

$y - 1 = 3(x - 2)$

$\underline{\underline{y = 3x - 5}}$

Calculus

Differentiation by rule

y	$\frac{dy}{dx}$
x^n	nx^{n-1}
ax^n	anx^{n-1}
ax	a
a	0
$u+v-w$	$\frac{du}{dx}+\frac{dv}{dx}-\frac{dw}{dx}$

Examples

$$\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\tfrac{1}{2}x^{-\frac{1}{2}}=\underline{\frac{1}{2\sqrt{x}}}$$

$$\frac{d}{dx}\left(\frac{4}{x}\right)=\frac{d}{dx}\left(4x^{-1}\right)=-4x^{-2}=\underline{-\frac{4}{x^2}}$$

$$\frac{d}{dx}\left(\frac{x}{2}\right)=\frac{d}{dx}\left(\tfrac{1}{2}x\right)=\underline{\frac{1}{2}}$$

$$\frac{d}{dx}(10)=0$$

$$\frac{d}{dx}\left(3x^2-x-5\right)=\underline{6x-1}$$

Vocabulary and more notation

$\frac{dy}{dx}$ is the derivative of *y* (with respect to *x*)

$\frac{dy}{dx}$ is the differential coefficient of *y* (with respect to *x*).

Example $\quad y = x^3 - 4x^2 + 3x - 1$

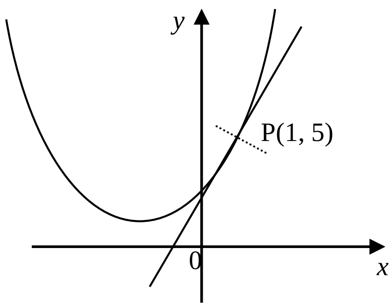
$$\frac{dy}{dx} = 3x^2 - 8x + 3$$

Example $\quad f(x)=\frac{x^2-2}{\sqrt{x}}=\frac{x^2}{\sqrt{x}}-\frac{2}{\sqrt{x}}=x^{\frac{3}{2}}-2x^{-\frac{1}{2}}$

$$\therefore \quad f'(x) = \tfrac{3}{2}x^{\frac{1}{2}} - \left(-\tfrac{1}{2}\right)2x^{-\frac{3}{2}} = \tfrac{3}{2}x^{\frac{1}{2}} + \frac{1}{x^{\frac{3}{2}}} = \underline{\frac{3}{2}\sqrt{x} + \frac{1}{x\sqrt{x}}}$$

- **<u>The gradient of a curve</u>** at any point is given by the value of $\frac{dy}{dx}$ at that point.

<u>Example</u> Find the gradient at the point P(1, 5) on the graph of $y = x^2 + 2x + 2$. Hence find the equation of the tangent at P.



$y = x^2 + 2x + 2$

$\frac{dy}{dx} = 2x + 2$

$\therefore$ At P(1, 5) gradient = 4

<u>Tangent at P</u>

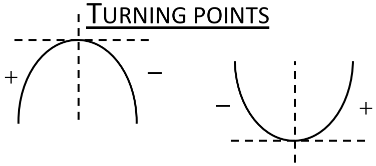
$y - 5 = 4(x - 1)$

$\therefore \underline{y = 4x + 1}$

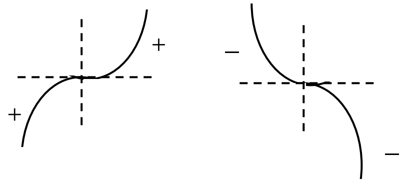
- **<u>Stationary points</u>** on the graph of a function are points where the gradient is zero.

STATIONARY POINTS

TURNING POINTS



POINTS OF INFLEXION



MAXIMUM POINT MINIMUM POINT TANGENT PASSING THROUGH THE CURVE

- **To obtain coordinates of a SP**. on the graph of $y = f(x)$

(i) Put $f'(x) = 0$ and solve for *x*.

(ii) If $x = a$ is a solution of (i) the SP will be $\{a, \quad f(a)\}$.

(iii) If $f''(a) > 0$ there will be a minimum point at $x = a$

If $f''(a) < 0$ there will be a maximum point at $x = a$

If $f''(a) = 0$ there could be max or min or inflexion so the second derivative rule fails. Investigate the gradient to the immediate left and right of the stationary point. (see the + and - signs on the diagrams in the previous section).

Example Find the stationary points on the graphs of

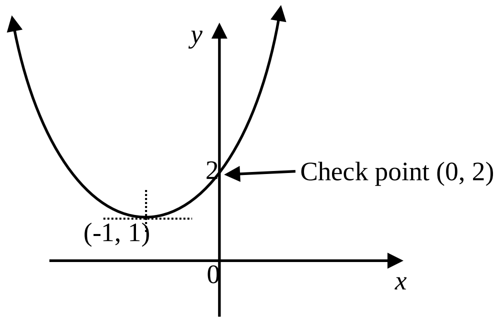
(i) $y = x^2 + 2x + 2$

(ii) $y = x^3 - 3x + 2$

and sketch the graphs.

(i) Here we have a quadratic function, which will have a true max or min.

$y = x^2 + 2x + 2$



$\frac{dy}{dx} = 2x + 2$

$\therefore$ SP at $2x + 2 = 0$

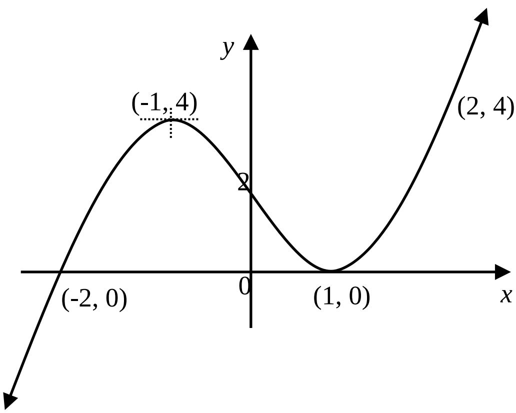
i.e. at $x = -1$

i.e. at $\underline{(-1, \quad 1)}$

$\frac{d^2y}{dx^2} = 2 > 0$

$\therefore$ SP is a minimum.

(ii) $y = x^3 - 3x + 2$



$\frac{dy}{dx} = 3x^2 - 3$

For SP $3x^2 - 3x = 0$

$\therefore \quad x^2 = 1$

$x = \pm 1$

$\therefore$ SPs at <u>(1, 0) (-1, 4)</u>

$\frac{d^2y}{dx^2} = 6x$

At (1, 0) $\frac{d^2y}{dx^2} = 6 > 0 \quad \therefore$ Min

At (-1, 4) $\frac{d^2y}{dx^2} = -6 < 0 \quad \therefore$ Max

Check points (0, 2) (2, 4) (-2, 0)

Note that the turning points are **<u>Local Max</u>** and **<u>Local Min</u>**

$$\frac{d^2y}{dx^2} = 6x$$

At (1, 0) $\frac{d^2y}{dx^2} = 6 > 0 \quad \therefore$ Min

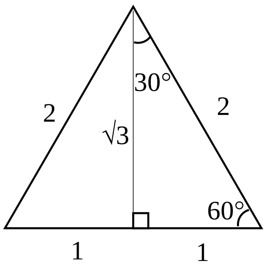
At (-1, 4) $\frac{d^2y}{dx^2} = -6 < 0 \quad \therefore$ Max

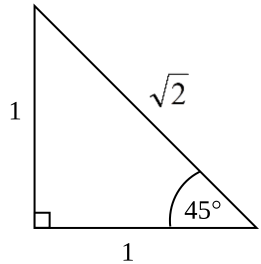
Check points (0, 2) (2, 4) (-2, 0)

Note that the turning points are **<u>Local Max</u>** and **<u>Local Min</u>**

Trigonometry

- **<u>Trig ratios for 30°, 60°, 45°</u>**





$\sin 30 = \cos 60 = \frac{1}{2}$

$\sin 60 = \cos 30 = \frac{\sqrt{3}}{2}$

$\tan 60 = \sqrt{3} \quad \tan 30 = \frac{1}{\sqrt{3}}$

$\sin 45 = \cos 45 = \frac{1}{\sqrt{2}}$

$\tan 45 = 1$

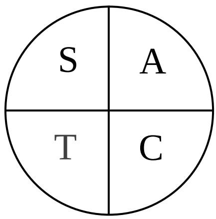
- **<u>Trig ratios for all angles</u>** NB the CAST DIAGRAM

For the sign of a trig ratio

All positive in first quadrant

Sine (only) in second quadrant

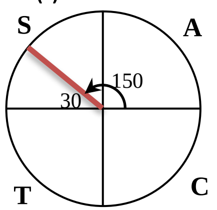
Etc…



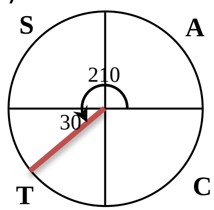
<u>Example</u> Without using a calculator find

(i) $\cos 150^\circ$ (ii) $\tan 210^\circ$ (iii) $\sin\left(-240^\circ\right)$

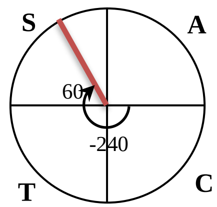
(i)



(ii)



(iii)



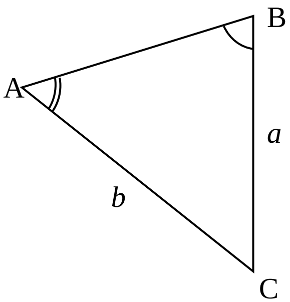
$$\cos 150^\circ = -\cos 30 = \underline{\underline{-\tfrac{\sqrt{3}}{2}}}$$

$$\tan 210^\circ = \tan 30 = \underline{\underline{\tfrac{1}{\sqrt{3}}}}$$

$$\sin\left(-240^\circ\right) = \sin 60 = \underline{\underline{\tfrac{\sqrt{3}}{2}}}$$

❑ **Trig of Scalene triangles**

<u>Sine rule</u>

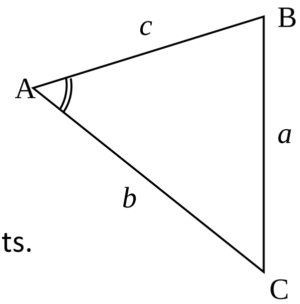


$$\frac{a}{\sin A} = \frac{b}{\sin B} = \left(\frac{c}{\sin C}\right)$$

Given AAS use it to find a second side

Given SSA use it to find a second angle (but take care to choose the angle size appropriately –it could be acute or obtuse).

Cosine rule



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

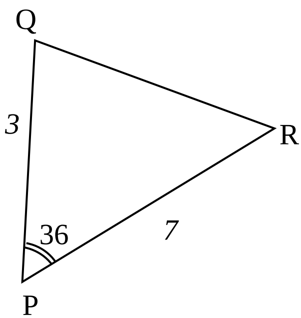
Both formulae with two more sets.

Given SAS use it to find the third side

Given SSS use it to find an angle (no possible ambiguity here).

Example   Triangle *PQR* has *PR* = 3cm, *QR* = 7cm and $Q\hat{P}R = 36^\circ$

Find (i) QR using the cosine rule and then (ii) $P\hat{Q}R$ using the sine rule.



(i) $QR^2 = 9 + 49 - 42\cos 36 = 24.021...$

$QR = 4.901.. \approx \underline{\underline{4.90}}$

(ii) $\dfrac{7}{\sin PQR} = \dfrac{4.901..}{\sin 36}$

$$\sin PQR = \frac{7\sin 36}{4.901..} = 0.8394...$$

$\therefore\ PQR = 57.086..$ or $PQR = 122.914..$

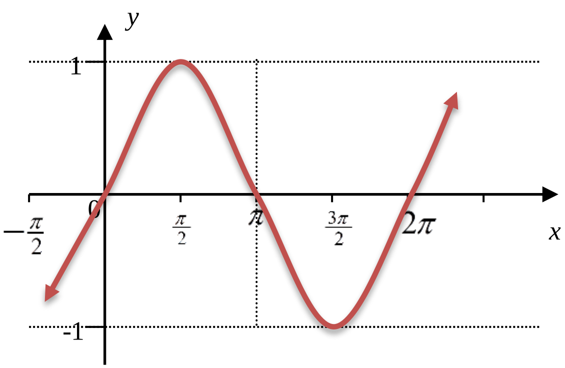
It can't be 57.08.. since R would be 86.92.. and would be the largest angle in the triangle, but R faces the smallest side so is the smallest angle. Hence $\underline{PQR = 122.91}$

- **Area "$\frac{1}{2}ab\sin C$" rule** given SAS

  Area of triangle = $\frac{1}{2}ab\sin C$

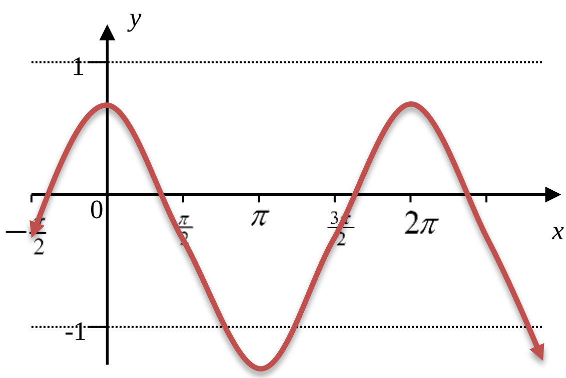
- **<u>Graphs of trig functions</u> (all periodic)**

1. Graph of $y = \sin x$



Period 360

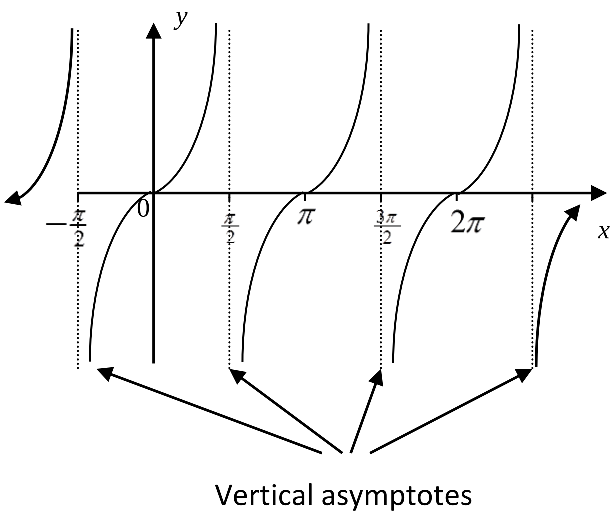
2. Graph of $y = \cos x$



Period 360

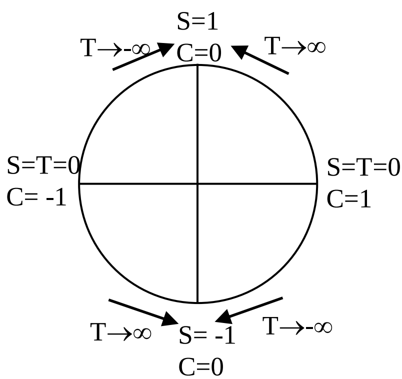
3. Graph of $y = \tan x$

Period 180



- **<u>Boundary values of trig ratios</u>**

Verify these from graphs



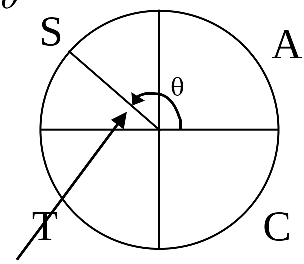
Two important trig identities

$$\boxed{\frac{\sin\theta}{\cos\theta} = \tan\theta} \qquad \boxed{\sin^2\theta + \cos^2\theta = 1}$$

<u>Example</u> Given θ is obtuse and $\sin\theta = \frac{8}{17}$ find the values of $\cos\theta$ and $\tan\theta$.

$\sin^2\theta + \cos^2\theta = 1$

$$\therefore\ \cos^2\theta = 1 - \sin^2\theta$$
$$= 1 - \tfrac{64}{289}$$
$$= \tfrac{225}{289}$$
$$\therefore\ \cos\theta = -\tfrac{15}{17}$$



$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \therefore\ \tan\theta = \frac{\frac{8}{17}}{-\frac{15}{17}} = -\tfrac{8}{15}$$

<u>NB</u> Learn how to rearrange the identities

$$\sin\theta = \cos\theta\tan\theta \qquad \cos\theta = \frac{\sin\theta}{\tan\theta}$$

$$\cos^2\theta = 1 - \sin^2\theta \qquad \sin^2\theta = 1 - \cos^2\theta$$

Trig equations

<u>Trig equations</u> Remember that from your calculator $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ give the ***principal value*** (p.v.)

<u>Example</u> Solve the equations

(i) $\tan\theta = -1.5$ for $0^\circ < \theta < 360^\circ$

(ii) $\sin 2\theta = 0.5$ for $-180^\circ < \theta < 180^\circ$

(iii) $2\cos^2\theta = 1 - \sin\theta$ for $0^\circ < \theta < 360^\circ$

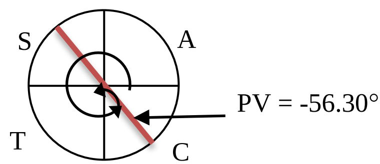
(iv) $2\sin^2\theta = \sin\theta\cos\theta$ for $0^\circ < \theta < 360^\circ$

(v) $\sin(\theta - 80) = \dfrac{\sqrt{3}}{2}$ for $-180^\circ < \theta < 180^\circ$

(i)

$\tan\theta = -1.5$

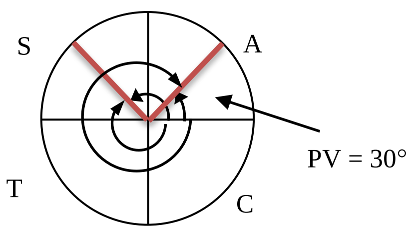
$\underline{\theta \approx 124^\circ,\ \ 304^\circ}$



(ii) $\sin 2\theta = 0.5$ .....first solve for 2θ for $-360^\circ < \theta < 360^\circ$

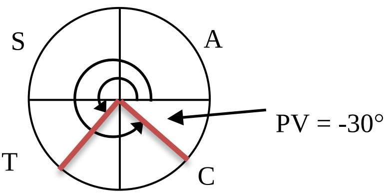
$2\theta = 30,\ 150;\ -210,\ -330$

$\therefore \underline{\theta = 15^\circ,\ 75^\circ;\ -105^\circ,\ -165^\circ}$



(iii) (In this example, use $\cos^2\theta = 1-\sin^2\theta$)

$$2\cos^2\theta = 1-\sin\theta$$
$$2(1-\sin^2\theta) = 1-\sin\theta$$
$$2-2\sin^2\theta = 1-\sin\theta$$
$$2\sin^2\theta - \sin\theta - 1 = 0$$
$$(\sin\theta - 1)(2\sin\theta + 1) = 0$$



$\therefore$ $\sin\theta = 1$ or $\sin\theta = -\frac{1}{2}$

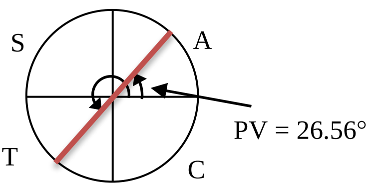
$\theta = 90^\circ$ $\qquad$ $\theta = 210^\circ,\ 330^\circ$

$\therefore\ \underline{\theta = 90^\circ,\ 210^\circ,\ 330^\circ}$

(iv) Don't cancel out $\sin\theta$. Bring to LHS and factorise

$$2\sin^2\theta = \sin\theta\cos\theta$$
$$2\sin^2\theta - \sin\theta\cos\theta = 0$$
$$\sin\theta(2\sin\theta - \cos\theta) = 0$$

$\therefore$ $\sin\theta = 0$ or $2\sin\theta = \cos\theta$

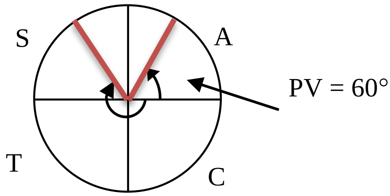


$\theta = 0^\circ,\ 180^\circ$ $\qquad$ $\dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{2}$

$$\tan\theta = \frac{1}{2}$$
$$\theta \approx 27^\circ,\ 207^\circ$$

$\therefore\ \underline{\theta = 0^\circ,\ 180^\circ,\ 27^\circ,}\ 207^\circ$

(v) $\sin(\theta - 80) = \frac{\sqrt{3}}{2}$ solve first for $-260^\circ < \theta < 100^\circ$



$(\theta - 80) = 60^\circ, \ -240^\circ$

$\therefore \underline{\theta = 140^\circ, \ -160^\circ}$

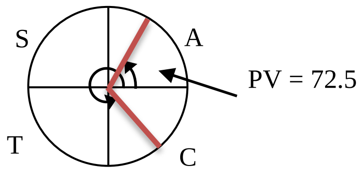
<u>Example</u> Solve the following equations

(i) $\cos x = 0.3$ for $0 < x < 360$, answers correct to 2d.p.

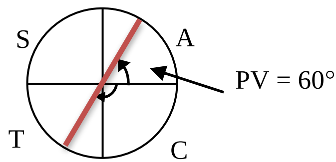
(ii) $\tan \frac{x}{2} = \sqrt{3}$ for $0 < x < 360$, answers in exact form

(i) $\cos x = 0.3$.

$\underline{x = 75.5, \ 287.5}$



(ii) $\tan \frac{x}{2} = \sqrt{3}$ ... solve first for $0 < x < 360$



$\frac{x}{2} = 60^\circ, 240^\circ$

$\therefore \underline{x = 120^\circ}$

Matrices

1. Multiplying matrices

In general

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

A 2x2 matrix multiplied by a 2x1 gives a 2x1 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{pmatrix}$$

A 2 by 2 matrix multiplied by a 2 by 2 gives a 2 by 2 matrix

<u>Example</u> If A = $\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}$ and B = $\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$

Work out (i) AB (ii) AA

(i) AB = $\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$

= $\begin{pmatrix} -5 & 9 \\ 3 & 3 \end{pmatrix}$

Worked out by the sum
-1x2 +3x-1

(ii) AA = $\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}$

= $\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$

Worked out by the sum
2x3 +1x1

<u>Example</u> Find the values of a and b when

$$\begin{pmatrix} -2 & -3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

From the first row $-2a - 3b = 5$

From the 2nd row $3a + 5b = -2$

Now solve simultaneously

X5 $-10a - 15b = 25$

X3 $9a + 15b = -6$

$\therefore$ $\underline{a = -19}$

And $-57 + 5b = -2$

$5b = 55$

$\underline{b = 11}$

2. Multiplying a matrix by a number.

<u>Example</u>

Given $A = \begin{pmatrix} -1 & 0 \\ 5 & 3 \end{pmatrix}$ work out $3A$

$$3A = 3\begin{pmatrix} -1 & 0 \\ 5 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 0 \\ 15 & 9 \end{pmatrix}$$

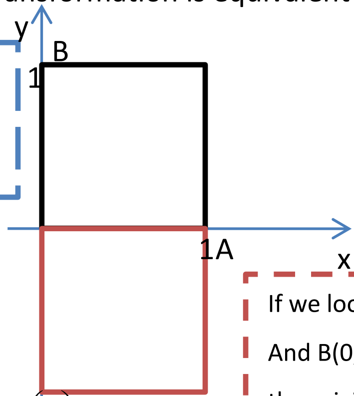
Each element in the matrix is multiplied by the constant 3.

3. Using matrices for transformations

<u>Example</u>

Which transformation is equivalent to a reflection in the *x*-axis?

Making use of the unit square reduces the amount of work required.



If we look at the movement of A(1, 0)

And B(0, 1) when the transformation is about

the origin then we can obtain the matrix

$$A\begin{pmatrix}1\\0\end{pmatrix} \rightarrow A'\begin{pmatrix}1\\0\end{pmatrix}$$

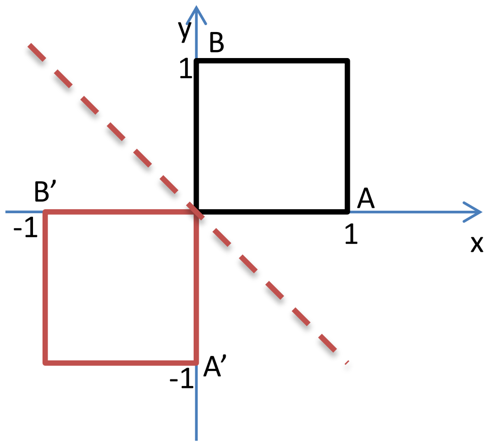
$$B\begin{pmatrix}0\\1\end{pmatrix} \rightarrow B'\begin{pmatrix}0\\-1\end{pmatrix}$$

Hence $$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{array}{c} \begin{array}{cc} A' & B' \end{array} \\ \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \end{array} \begin{pmatrix}x\\y\end{pmatrix}$$

<u>Example</u>

Which transformation is defined by the matrix $\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$?

Again drawing the unit square and looking at where A(1,0) and B(0, 1) moves to will help identify this matrix.



$$A\begin{pmatrix}1\\0\end{pmatrix} \rightarrow A'\begin{pmatrix}0\\-1\end{pmatrix}$$

$$B\begin{pmatrix}0\\1\end{pmatrix} \rightarrow B'\begin{pmatrix}-1\\0\end{pmatrix}$$

$\therefore$ A reflection in the line $y = -x$

4. <u>Combinations of transformations</u>.

M represents the transformation given by $\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$

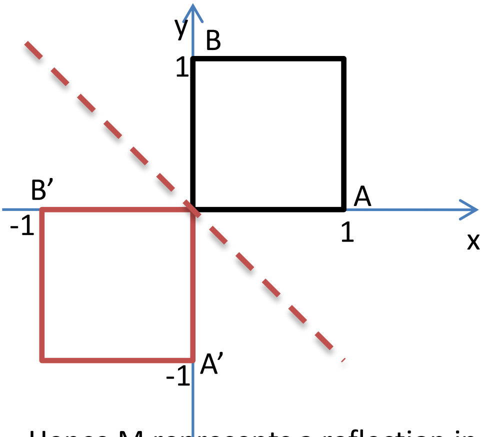
N represents the transformation given by $\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$

a) Describe matrix M.

b) Describe matrix N.

c) Find the single transformation for the transformation MN and its description.

a)

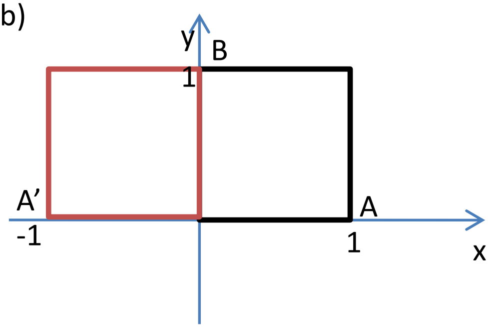


$$A\begin{pmatrix}1\\0\end{pmatrix} \to A'\begin{pmatrix}0\\-1\end{pmatrix}$$

$$B\begin{pmatrix}0\\1\end{pmatrix} \to B'\begin{pmatrix}-1\\0\end{pmatrix}$$

Hence M represents a reflection in $y = -x$

b)



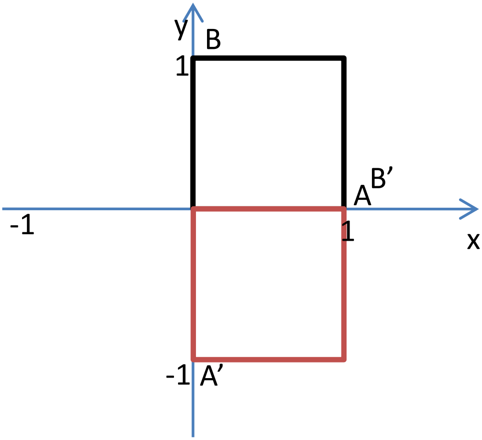
$$A\begin{pmatrix}1\\0\end{pmatrix} \to A'\begin{pmatrix}-1\\0\end{pmatrix}$$

$$B\begin{pmatrix}0\\1\end{pmatrix} \to B'\begin{pmatrix}0\\1\end{pmatrix}$$

Hence N represents a reflection in the y axis

c) $MN = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$

$$MN = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$$



$$A\begin{pmatrix}1\\0\end{pmatrix}\rightarrow A'\begin{pmatrix}0\\-1\end{pmatrix}$$

$$B\begin{pmatrix}0\\1\end{pmatrix}\rightarrow B'\begin{pmatrix}1\\0\end{pmatrix}$$

Hence MN represents a rotation of $90°$ clockwise centre (0,0)

Notes

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